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Well-posedness and solution structure of dual-phase-lagging heat conduction

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Abstract

The dual-phase-lagging heat conduction equation is shown to be well-posed in a finite 1D region under Dirichlet, Neumann or Robin boundary conditions. Two solution structure theorems are developed for dual-phase-lagging heat conduction equations under linear boundary conditions. These theorems express contributions (to the temperature field) of the initial temperature distribution and the source term by that of the initial time-rate change of the temperature. This reveals the structure of the temperature field and considerably simplifies the development of solutions of dual-phase-lagging heat conduction equations. © 2001 Elsevier Science Ltd. All rights reserved.

1. Introduction

By lumping microstructural effects into delayed temporal responses in the macroscopic formulation, Tzou [1] proposed a dual-phase-lagging constitutive equation for heat conduction, relating the temperature gradient ∇T at a material point p and time $t+\tau_T$ to the heat flux density vector $\mathbf{q}$ at p and time $t+\tau_q$ through material thermal conductivity k

$$\mathbf{q}(p, t+\tau_q) = -k\nabla T(p, t+\tau_T). \tag{1}$$

Two delay times τ_T and τ_q are regarded as intrinsic thermal or structural properties of the material. The former is due to the microstructural interactions such as phonon–electron interaction or phonon scattering, and is termed as the phase-lag of the temperature gradient. The latter is, on the other hand, interpreted as the relaxation time accounting for the fast-transient effects of thermal inertia, and is named as the phase-lag of the heat flux.

Expanding ∇T and $\mathbf{q}$ with respect to time t by Taylor's series and retaining only the first-order terms in τ_T and τ_q, we obtain a linear version of (1) at the point p and time t [1]

$$\mathbf{q} + \tau_q \frac{\partial \mathbf{q}}{\partial t} = -k\left[\nabla T + \tau_T \frac{\partial}{\partial t}(\nabla T)\right] \tag{2}$$

which is known as the Jeffreys-type constitutive equation of heat flux [2]. Eliminating $\mathbf{q}$ from (2)and the classical energy equation leads to the dual-phase-lagging heat conduction equation [1] that reads, if all thermophysical material properties are assumed to be constant,

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_q}{\alpha}\frac{\partial^2 T}{\partial t^2} = \Delta T + \tau_T \frac{\partial}{\partial t}(\Delta T) + f(P, t), \tag{3}$$

where α is the thermal diffusivity of the material, Δ the Laplacian, and f stands for terms from internal heat sources. The dual-phase-lagging heat conduction equation forms a generalized, unified equation that reduces

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[1] For the heat conduction involving heat flux-specified boundary conditions, it is more convenient to use the dual-phase-lagging heat conduction equation in terms of the heat flux $\mathbf{q}$ or the heat flux potential ϕ defined by $\mathbf{q} = \nabla\phi$. It can be obtained by eliminating T from Eq. (2) and the classical energy equation. The ϕ-version heat conduction equation has exactly the same structure as its T-version (3). The 1D $\mathbf{q}$-version heat conduction equation is also of the same structure as its T-version. A mixed formulation for both $\mathbf{q}$ and T directly by two-coupled energy and constitutive equations is more general than Eq. (3) in view of the applications of the dual-phase-lagging model. The readers are referred to Tzou [3, pp. 30–34] for details.

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Nomenclature

l	length
$\mathbf{q}$	heat flux vector
p	material point
T	temperature
t	time
x	coordinate
k	thermal conductivity

Greek symbols

α	thermal diffusivity
ρ	density
τ	thermal relaxation time
τ_q	phase-lag of the heat flux vector
τ_T	phase-lag of the temperature gradient
∇	gradient
Δ	Laplacian

to the classical parabolic heat conduction equation when $\tau_T = \tau_q = 0$, the hyperbolic heat conduction equation when $\tau_T = 0$ and $\tau_q = \tau$ with τ as the relaxation time defined by Chester [4], the energy equation in the phonon scattering model [2,5] when

$$\alpha = \frac{\tau_R c^2}{3}, \quad \tau_T = \frac{9}{5}\tau_N, \quad \tau_q = \tau_R,$$

and the energy equation in the phonon–electron interaction model [6–8] when

$$\alpha = \frac{k}{c_e + c_l}, \quad \tau_T = \frac{c_l}{G}, \quad \tau_q = \frac{1}{G}\left[\frac{1}{c_e} + \frac{1}{c_l}\right]^{-1}.$$

In the phonon scattering model, c is the average speed of phonons (sound speed), τ_R the relaxation time for the umklapp process in which momentum is lost from the phonon system, and τ_N is the relaxation time for normal processes in which momentum is conserved in the phonon system. In the phonon–electron interaction model, k is the thermal conductivity of the electron gas, G the phonon–electron coupling factor, and c_e and c_l are the heat capacity of the electron gas and the metal lattice, respectively. This, with the rapid growth of microscale heat conduction of high-rate heat flux, has given rise to the research effort on solutions of dual-phase-lagging heat conduction equations. The solutions of 1D heat conduction under some specific initial and boundary conditions were developed in [1,3,9–13]. Wang and Zhou [14] developed methods of measuring τ_T and τ_q and obtained analytical solutions for regular 1D, 2D and 3D heat conduction domains under essentially arbitrary initial and boundary conditions. While the dual-phase-lagging model yields a better prediction of microscale heat conduction and is admissible within the framework of the second law of the extended irreversible thermodynamics [3], some fundamental issues such as well-posedness and solution structure have been left unaddressed.

The present work aims to examine the well-posedness and solution structure of initial–boundary value problems for dual-phase-lagging heat conduction equations. In particular, we establish the existence, uniqueness and stability of the solution with respect to initial conditions for the dual-phase-lagging heat conduction in a finite 1D region under homogeneous Dirichlet, Neumann or Robin boundary conditions. We also extend the two theorems of solution structure for the hyperbolic heat-conduction in [15] to the dual-phase-lagging heat conduction under linear homogeneous boundary conditions. Such theorems relate contributions (to the temperature field) of the initial temperature distribution and the source term to that of the initial time-rate change of the temperature. This reveals the structure of the temperature field and significantly simplifies the development of solutions of dual-phase-lagging heat conduction equations.

2. Well-posedness

In this section, we consider the well-posedness of the 1D initial–boundary value problem with homogeneous Dirichlet (first kind), Neumann (second kind) and Robin (third kind) boundary conditions

$$\begin{cases} \frac{1}{\alpha}T_t(x,t) + \frac{\tau_q}{\alpha}T_{tt}(x,t) = T_{xx}(x,t) \\ \quad +\tau_T T_{txx}(x,t), \\ (0,l)\times(0,+\infty), \\ \{-k_1T_x(x,t) + h_1T(x,t)\}|_{x=0} = 0, \\ \{k_2T_x(x,t) + h_2T(x,t)\}|_{x=l} = 0, \\ T(x,0) = \phi(x), T_t(x,0) = \psi(x), \end{cases} \tag{4}$$

where k_1, k_2, h_1 and h_2 are the nonnegative real constants and satisfy

$$k_1 + h_1 \neq 0 \tag{5}$$

and

$$k_2 + h_2 \neq 0. \tag{6}$$

The readers are referred to Tzou [3] and Fournier and Boccara [16] for physical implications and limitations of three boundary conditions.

If all combinations of the boundary conditions of the first, second and third kinds are considered, for a finite region $0 \leqslant x \leqslant l$, there exist nine combinations of boundary conditions. We detail the well-posedness for the case with the third (Robin) boundary condition at both $x = 0$ and $x = l$, i.e., nonzero finite k_1, h_1, k_2 and h_2. The results for the remaining eight combinations are easily obtained by a similar approach and choosing the values of H_1 and H_2 as zero, finite or infinite, and are listed in the tables if appropriate. Here H_1 and H_2 are defined by

$$H_1 = \frac{h_1}{k_1} \tag{7}$$

and

$$H_2 = \frac{h_2}{k_2}. \tag{8}$$

2.1. Existence

For the existence, we use the separation of variables to find a solution of (4). Assuming separation of the variables in the form

$$T(x,t) = X(x)\Gamma(t). \tag{9}$$

A substitution of (9) into (4) leads to

$$X(x)\left[\frac{1}{\alpha}\Gamma'(t) + \frac{\tau_q}{\alpha}\Gamma''(t)\right] = X''(x)[\Gamma(t) + \tau_T\Gamma'(t)]$$

which becomes, after dividing by $X(x)[\Gamma(t) + \tau_T\Gamma'(t)]$, [2]

$$\frac{\frac{1}{\alpha}\Gamma'(t) + \frac{\tau_q}{\alpha}\Gamma''(t)}{\Gamma(t) + \tau_T\Gamma'(t)} = \frac{X''(x)}{X(x)},$$

where the primes on the functions X and Γ represent differentiation with respect to the only variable present. Therefore, we have the separation equation for the temporal variable $\Gamma(t)$

$$\tau_q\Gamma''(t) + (1 + \alpha\tau_T\lambda)\Gamma'(t) + \alpha\lambda\Gamma(t) = 0 \tag{10}$$

and the homogeneous system for the spatial variable $X(x)$

$$X''(x) + \lambda X(x) = 0, \quad (0, l), \tag{11}$$

$$-k_1X'(0) + h_1X(0) = 0, \tag{12}$$

$$k_2X'(l) + h_2X(l) = 0, \tag{13}$$

where λ is the separation constant.

Integrating Eq. (11) from $x = 0$ to $x = l$ after multiplying $X(x)$ yields

$$\int_0^l X(x)X''(x)\,\mathrm{d}x + \lambda\int_0^l X^2(x)\,\mathrm{d}x = 0 \tag{14}$$

which becomes, by making use of integration by parts to $\int_0^l X(x)X''(x)\,\mathrm{d}x$,

$$\lambda\int_0^l X^2(x)\,\mathrm{d}x = -X(x)X'(x)|_0^l + \int_0^l [X''(x)]^2\,\mathrm{d}x. \tag{15}$$

By (12) and (13)

$$X'(0) = H_1X(0),$$

$$X'(l) = -H_2X(l).$$

Therefore, (14) leads to

$$\lambda\int_0^l X^2(x)\,\mathrm{d}x = H_1X^2(0) + H_2X^2(l) + \int_0^l [X''(x)]^2\,\mathrm{d}x$$

which implies

$$\lambda \geqslant 0. \tag{16}$$

If $\lambda = 0$, however,

$$X(x) = c_1x + c_2 \tag{17}$$

with c_1 and c_2 as two constants. The two boundary conditions (12) and (13) thus lead to

$$c_1 - H_1c_2 = 0,$$

$$c_1(1 + H_2l) + H_2c_2 = 0$$

which yield $c_1 = c_2 = 0$ by noting that $H_1 > 0$ and $H_2 > 0$. We therefore arrive at a trivial solution when $\lambda = 0$. Hence,

$$\lambda = \beta^2 > 0. \tag{18}$$

The general solution of (11) is thus

$$X(x) = a\,\cos(\beta x) + b\,\sin(\beta x),$$

where a and b are two constants. The two boundary conditions (12) and (13) thus yield

$$\beta b - H_1a = 0,$$

$$-\beta a\,\sin(\beta l) + \beta b\,\cos(\beta l) + H_2a\,\cos(\beta l) + H_1b\,\sin(\beta l) = 0$$

which lead to

$$\mathrm{ctg}(l\beta) = \frac{1}{l(H_1 + H_2)}\left[l\beta - \frac{(lH_2)^2}{l\beta}\right]. \tag{19}$$

[2] For a nontrivial solution, both $X(x)$ and $\Gamma(t)$ cannot be vanished for all x or t. $\Gamma(t) + \tau_T\Gamma'(t)$ cannot be trivial neither. Otherwise, $\Gamma(t)$ must satisfy *both*

$$\Gamma(t) + \tau_T\Gamma'(t) = 0$$

and

$$\frac{1}{\alpha}\Gamma'(t) + \frac{\tau_q}{\alpha}\Gamma''(t) = 0$$

which have no solution. Dividing by $X(x)\Gamma(t)$ as in the classical separation of variables would lead to

$$\frac{1}{\alpha}\frac{\Gamma'(t)}{\Gamma(t)} + \frac{\tau_q}{\alpha}\frac{\Gamma''(t)}{\Gamma(t)} = \frac{X''(x)}{X(x)} + \tau_T\frac{X''(x)\Gamma'(t)}{X(x)\Gamma(t)}$$

whose last term is nonseparable. This seems to conclude in the literature that the method of separation of variables fails to apply (see, for example, [3, p. 48]).

Let

$$f(x) = \operatorname{ctg} x - \frac{1}{l(H_1+H_2)}\left[x - \frac{(lH_2)^2}{x}\right], \tag{20}$$

$l\beta$ is thus the zero point of $f(x)$. As $f(x)$ is an odd function and $\lambda = \beta^2$, we only need the positive zero points of $f(x)$. Let β_m be the mth positive zero point of $f(x)$. Hence, we have the eigenvalues,

$$\lambda_m = \left(\frac{\beta_m}{l}\right)^2, \quad m = 1, 2, \ldots \tag{21}$$

Without taking account of arbitrary constants a and b, the eigenfunctions can be written as

$$\frac{\beta_m}{lH_1}\cos\left(\frac{\beta_m x}{l}\right) + \sin\left(\frac{\beta_m x}{l}\right) = \sqrt{1+\left(\frac{\beta_m}{lH_1}\right)^2}\sin\left(\frac{\beta_m x}{l}+\phi_m\right), \tag{22}$$

where

$$\operatorname{tg}\phi_m = \frac{\beta_m}{lH_1}. \tag{23}$$

Finally, we can write the eigenfunction $X_m(x)$ as

$$X_m(x) = \sin\left(\frac{\beta_m x}{l}+\phi_m\right) \tag{24}$$

without taking account of arbitrary constants.

Consider now Eq. (10) that reads, after substituting λ_m in Eq. (21),

$$\tau_q \Gamma''_m(t) + (1+\alpha\tau_T\beta_m^2)\Gamma'_m(t) + \alpha\beta_m^2\Gamma_m(t) = 0. \tag{25}$$

Two characteristic roots of the auxiliary equation of (25) are

$$r_{1,2} = v_m \pm \mathrm{i}\mu_m, \tag{26}$$

where

$$v_m = -\frac{1+\alpha\tau_T\beta_m^2}{2\tau_q} \tag{27}$$

and

$$\mu_m = \frac{\sqrt{4\alpha\tau_q\beta_m^2 - (1+\alpha\tau_T\beta_m^2)^2}}{2\tau_q}. \tag{28}$$

Note that the μ_m can take real or imaginary values. Therefore,

$$\Gamma_m(t) = \mathrm{e}^{v_m t}[A_m\cos(\mu_m t) + B_m\,\underline{\sin}(\mu_m t)], \tag{29}$$

where A_m and B_m are constants, and

$$\underline{\sin}(\mu_m t) = \begin{cases}\sin(\mu_m t) & \text{if } \mu_m \neq 0,\\ t & \text{if } \mu_m = 0.\end{cases} \tag{30}$$

Applying the principle of superposition to (4) yields

$$T(x,t) = \sum_{m=1}^{\infty}\mathrm{e}^{v_m t}[A_m\cos(\mu_m t) + B_m\,\underline{\sin}(\mu_m t)]X_m(x). \tag{31}$$

Applying the first initial condition $T(x,0) = \phi(x)$ leads to

$$\sum_{m=1}^{\infty}A_m X_m(x) = \phi(x) \tag{32}$$

which requires, by the Sturm–Liouvill theory that $X_m(x)$ $(m = 1, 2, \ldots)$ forms a complete orthogonal set in $0 \leqslant x \leqslant l$,

$$A_m = \frac{1}{M_m}\int_0^l \phi(x)X_m(x)\,\mathrm{d}x, \quad m = 1, 2, \ldots \tag{33}$$

Here,

$$M_m = \int_0^l X_m^2(x)\,\mathrm{d}x = \int_0^l \sin^2\left(\frac{\beta_m x}{l}+\phi_m\right)\mathrm{d}x = \frac{l}{2}\left[1 - \frac{\sin\beta_m}{\beta_m}\cos(\beta_m + 2\phi_m)\right]. \tag{34}$$

Also,

$$T_t(x,t) = \sum_{m=1}^{+\infty}\{v_m\mathrm{e}^{v_m t}[A_m\cos(\mu_m t) + B_m\,\underline{\sin}(\mu_m t)] + \mathrm{e}^{v_m t}[-A_m\mu_m\sin(\mu_m t) + B_m\,\underline{\mu}_m\cos(\mu_m t)]\}X_m(x),$$

where

$$\underline{\mu}_m = \begin{cases}\mu_m & \text{if } \mu_m \neq 0,\\ 1 & \text{if } \mu_m = 0.\end{cases} \tag{35}$$

Applying the second initial condition $T_t(x,0) = \psi(x)$ yields

$$\sum_{m=1}^{+\infty}(v_m A_m + B_m\underline{\mu}_m)X_m(x) = \psi(x) \tag{36}$$

which requires, by the Sturm–Liouvill theory,

$$v_m A_m + B_m\underline{\mu}_m = \frac{1}{M_m}\int_0^l \psi(x)X_m(x)\,\mathrm{d}x$$

i.e.,

$$B_m = \frac{1}{M_m\underline{\mu}_m}\int_0^l \psi(x)X_m(x)\,\mathrm{d}x - \frac{v_m A_m}{\underline{\mu}_m}. \tag{37}$$

Therefore, we have found a solution of (4)

$$\begin{cases} T(x,t) = \sum_{m=1}^{\infty}\mathrm{e}^{v_m t}(A_m\cos\mu_m t + B_m\,\underline{\sin}\mu_m t)X_m(x),\\ A_m = \frac{1}{M_m}\int_0^l \phi(x)X_m(x)\,\mathrm{d}x,\\ B_m = \frac{1}{M_m\underline{\mu}_m}\int_0^l \psi(x)X_m(x)\,\mathrm{d}x - \frac{v_m}{\underline{\mu}_m M_m}\int_0^l \phi(x)X_m(x)\,\mathrm{d}x,\end{cases} \tag{38}$$

with v_m, μ_m and $\underline{\mu}_m$ defined by Eqs. (27), (28) and (35), respectively.

Actually, the solution of (4) under the other eight boundary conditions can also be written in the form of (38). However, eigenvalues λ_m, eigenfunction $X_m(x)$ and normal square M_m are different for the nine combinations of boundary conditions (see Table 1).

2.2. An inequality

To establish the uniqueness and stability of solution of (4), we need first to develop an important inequality for (4). Once again, we detail the development for the case with the Robin boundary condition at both $x = 0$ and $x = l$, and list the final results for the remaining eight combinations of boundary conditions. In the process of deriving the inequality, a commonly used assumption is made that the order of differentiation is interchangeable for some high-order partial derivatives of T with respect to t and x. While the continuity of the associated high-order partial derivatives forms the sufficient condition for such an interchange, it is *not* the necessary condition. Therefore, the interchange of the order of differentiation could still be valid even in the region where some high-order partial derivatives of T could be discontinuous (the dissipating or damping feature of dual-phase-lagging heat-conduction equations would hinder the appearance of such discontinuity). As both the necessary and sufficient conditions are unavailable in mathematics, it appears not possible, at the present, to state what are the conditions that T should possess in order to be able to interchange the order of differentiation.

Note that

$$\begin{aligned}\frac{\partial}{\partial t}(T+\tau_q T_t)^2 &= 2(T+\tau_q T_t)(T_t+\tau_q T_{tt})\\ &= 2\alpha(T+\tau_q T_t)(T_{xx}+\tau_T T_{txx})\end{aligned} \tag{39}$$

in which the heat-conduction equation in (4) has been used. Integrating (39) with respect to x from $x = 0$ to $x = l$ and using integration by parts lead to

$$\begin{aligned}&\int_0^l \frac{\partial}{\partial t}(T+\tau_q T_t)^2\,\mathrm{d}x\\ &= 2\alpha\int_0^l (T+\tau_q T_t)(T_{xx}+\tau_T T_{txx})\,\mathrm{d}x\\ &= 2\alpha\Big\{[T(l,t)+\tau_q T_t(l,t)][T_x(l,t)+\tau_T T_{tx}(l,t)]\\ &\quad -[T(0,t)+\tau_q T_t(0,t)][T_x(0,t)+\tau_T T_{tx}(0,t)]\\ &\quad -\int_0^L\left[T_x^2+\frac{1}{2}(\tau_T+\tau_q)\frac{\partial}{\partial t}T_x^2+\tau_T\tau_q T_{tx}^2\right]\mathrm{d}x\Big\}.\end{aligned} \tag{40}$$

Using two boundary conditions in (4), Eq. (40) can be rearranged to

$$\begin{aligned}&\alpha H_2(\tau_T+\tau_q)\frac{\partial}{\partial t}T^2(l,t)+\alpha H_1(\tau_T+\tau_q)\frac{\partial}{\partial t}T^2(0,t)\\ &\quad+\int_0^l\frac{\partial}{\partial t}\left[\left(T+\tau_q\frac{\partial T}{\partial t}\right)^2+\alpha(\tau_T+\tau_q)\left(\frac{\partial T}{\partial x}\right)^2\right]\mathrm{d}x\\ &= -2\alpha H_2 T^2(l,t)-2\alpha H_1 T^2(0,t)\\ &\quad-2\alpha\int_0^L\left(\frac{\partial T}{\partial x}\right)^2\mathrm{d}x-2\alpha\tau_T\tau_q H_2\left[\frac{\partial T(l,t)}{\partial t}\right]^2\\ &\quad-2\alpha\tau_T\tau_q H_1\left[\frac{\partial T(0,t)}{\partial t}\right]^2-2\alpha\tau_T\tau_q\int_0^l\left(\frac{\partial^2 T}{\partial x\partial t}\right)^2\mathrm{d}x\end{aligned} \tag{41}$$

which is negative semi-definite because $\alpha, \tau_T, \tau_q, H_1$ and H_2 are all not negative, i.e.,

$$\begin{aligned}&\alpha H_2(\tau_T+\tau_q)\frac{\partial}{\partial t}T^2(l,t)+\alpha H_1(\tau_T+\tau_q)\frac{\partial}{\partial t}T^2(0,t)\\ &\quad+\int_0^l\frac{\partial}{\partial t}\left[(T+\tau_q T_t)^2+\alpha(\tau_T+\tau_q)T_x^2\right]\mathrm{d}x\leqslant 0.\end{aligned} \tag{42}$$

Integrating (42) with respect to t from t_0 to t_1 ($t_1 \geqslant t_0$) yields an important inequality

$$g(t_1)\leqslant g(t_0),\quad \forall t_1\geq t_0, \tag{43}$$

where

$$\begin{aligned}g(t) &= \alpha(\tau_T+\tau_q)[H_1 T^2(0,t)+H_2 T^2(l,t)]\\ &\quad+\int_0^l\{[T(x,t)+\tau_q T_t(x,t)]^2+\alpha(\tau_T+\tau_q)T_x^2(x,t)\}\,\mathrm{d}x.\end{aligned} \tag{44}$$

The inequality for the other eight boundary conditions can also be written in the form of (43). However, the definition of $g(t)$ is different and is listed in Table 2.

2.3. Uniqueness

Suppose that $T_1(x,t)$ and $T_2(x,t)$ are two solutions of (4). The difference between them

$$W(x,t) = T_1(x,t) - T_2(x,t)$$

must be the solution of the initial–boundary value problem,

$$\begin{cases}\frac{1}{\alpha}W_t(x,t)+\frac{\tau_q}{\alpha}W_{tt}(x,t)=W_{xx}(x,t)\\ \quad+\tau_T W_{txx}(x,t),\quad (0,l)\times(0,+\infty),\\ -k_1W_x(0,t)+h_1W(0,t)=0,\\ k_2W_x(l,t)+h_2W(l,t)=0,\\ W(x,0)=0,\ W_t(x,0)=0.\end{cases} \tag{45}$$

For the case with the Robin condition at both $x = 0$ and l, an application of (43) to (45) yields, with $t_1 = t > 0$ and $t_0 = 0$,

Table 1
$\lambda_m, X_m(x)$ and M_m for nine combinations of boundary conditions

$x=0$	$x=l$	λ_m	$X_m(x)$	M_m	Notes
Dirichlet	Dirichlet	$\left(\frac{m\pi}{l}\right)^2$	$\sin\frac{m\pi x}{l}$	$\frac{l}{2}$	$m=1,2,\ldots$
Dirichlet	Neumann	$\left[\frac{(m+(1/2))\pi}{l}\right]^2$	$\sin\frac{(m+(1/2))\pi x}{l}$	$\frac{l}{2}$	$m=0,1,2,\ldots$
Dirichlet	Robin	$\left(\frac{\beta_m}{l}\right)^2$	$\sin\frac{\beta_m x}{l}$	$\frac{l}{2}\left(1-\frac{\sin 2\beta_m}{2\beta_m}\right)$	β_m $(m=1,2,\ldots)$ are positive zero points of $f_1(x)=\mathrm{tg}\,x+\frac{x}{lH_2}$
Neumann	Dirichlet	$\left[\frac{(m+(1/2))\pi}{l}\right]^2$	$\cos\frac{(m+(1/2))\pi x}{l}$	$\frac{l}{2}$	$m=0,1,2,\ldots$
Neumann	Neumann	$\left(\frac{m\pi}{l}\right)^2$	$\cos\frac{m\pi x}{l}$	$\frac{l}{2}$	$m=0,1,2,\ldots$
Neumann	Robin	$\left(\frac{\beta_m}{l}\right)^2$	$\cos\frac{\beta_m x}{l}$	$\frac{l}{2}\left(1+\frac{\sin 2\beta_m}{2\beta_m}\right)$	β_m $(m=1,2,\ldots)$ are positive zero points of $f_2(x)=\mathrm{tg}\,x-\frac{lH_2}{x}$
Robin	Dirichlet	$\left(\frac{\beta_m}{l}\right)^2$	$\sin\left(\frac{\beta_m x}{l}+\phi_m\right),\ \mathrm{tg}\ \phi_m=\frac{\beta_m}{lH_1}$	$\frac{l}{2}\left[1-\frac{\sin\beta_m}{\beta_m}\cos(\beta_m+2\phi_m)\right]$	β_m $(m=1,2,\ldots)$ are positive zero points of $f_3(x)=\mathrm{tg}\,x+\frac{x}{lH_1}$
Robin	Neumann	$\left(\frac{\beta_m}{l}\right)^2$	$\sin\left(\frac{\beta_m x}{l}+\phi_m\right),\ \mathrm{tg}\ \phi_m=\frac{\beta_m}{lH_1}$	$\frac{l}{2}\left[1-\frac{\sin\beta_m}{\beta_m}\cos(\beta_m+2\phi_m)\right]$	$\beta_m(m=1,2,\ldots)$ are positive zero points of $f_4(x)=\mathrm{tg}\,x-\frac{lH_1}{x}$
Robin	Robin	$\left(\frac{\beta_m}{l}\right)^2$	$\sin\left(\frac{\beta_m x}{l}+\phi_m\right),\ \mathrm{tg}\ \phi_m=\frac{\beta_m}{lH_1}$	$\frac{l}{2}\left[1-\frac{\sin\beta_m}{\beta_m}\cos(\beta_m+2\phi_m)\right]$	β_m $(m=1,2,\ldots)$ are positive zero points of $f_5(x)=\mathrm{ctg}\,x-\frac{1}{l(H_1+H_2)}\left[x-\frac{(lH_2)^2}{x}\right]$

Table 2
$g(t)$ for nine combinations of boundary conditions

$x=0$	$x=l$	$g(t)$
Dirichlet	Dirichlet	$\int_0^l\{[T(x,t)+\tau_q T_t(x,t)]^2+\alpha(\tau_T+\tau_q)T_x^2(x,t)\}\,\mathrm{d}x$
Dirichlet	Neumann	$\int_0^l\{[T(x,t)+\tau_q T_t(x,t)]^2+\alpha(\tau_T+\tau_q)T_x^2(x,t)\}\,\mathrm{d}x$
Dirichlet	Robin	$\alpha(\tau_T+\tau_q)H_2T^2(l,t)+\int_0^l\{[T(x,t)+\tau_q T_t(x,t)]^2+\alpha(\tau_T+\tau_q)T_x^2(x,t)\}\,\mathrm{d}x$
Neumann	Dirichlet	$\int_0^l\left\{[T(x,t)+\tau_q T_t(x,t)]^2+\alpha(\tau_T+\tau_q)T_x^2(x,t)\right\}\mathrm{d}x$
Neumann	Neumann	$\int_0^l\{[T(x,t)+\tau_q T_t(x,t)]^2+\alpha(\tau_T+\tau_q)T_x^2(x,t)\}\mathrm{d}x$
Neumann	Robin	$\int_0^l\{[T(x,t)+\tau_q T_t(x,t)]^2+\alpha(\tau_T+\tau_q)T_x^2(x,t)\}\,\mathrm{d}x$
Robin	Dirichlet	$\alpha(\tau_T+\tau_q)H_1T^2(0,t)+\int_0^l\{[T(x,t)+\tau_q T_t(x,t)]^2+\alpha(\tau_T+\tau_q)T_x^2(x,t)\}\,\mathrm{d}x$
Robin	Neumann	$\int_0^l\{[T(x,t)+\tau_q T_t(x,t)]^2+\alpha(\tau_T+\tau_q)T_x^2(x,t)\}\,\mathrm{d}x$
Robin	Robin	$\alpha(\tau_T+\tau_q)[H_1T^2(0,t)+H_2T^2(l,t)]+\int_0^l\{[T(x,t)+\tau_q T_t(x,t)]^2+\alpha(\tau_T+\tau_q)T_x^2(x,t)\}\,\mathrm{d}x$

$$
\begin{aligned}
&\alpha(\tau_T+\tau_q)[H_1W^2(0,t)+H_2W^2(l,t)]+\int_0^l\{[W(x,t)\\
&\quad+\tau_q W_t(x,t)]^2+\alpha(\tau_T+\tau_q)W_x^2(x,t)\}\,\mathrm{d}x\\
&\quad\leqslant\alpha(\tau_T+\tau_q)[H_1W^2(0,0)+H_2W^2(l,0)]+\int_0^l\{[W(x,0)\\
&\quad+\tau_q W_t(x,0)]^2+\alpha(\tau_T+\tau_q)W_x^2(x,0)\}\,\mathrm{d}x=0
\end{aligned}
\tag{46}
$$

by using the initial conditions in (45). This requires, as $\alpha, \tau_T, \tau_q, H_1$ and H_2 are all positive definite,

$$W_x(x,t)=0 \tag{47}$$

and

$$W(x,t)+\tau_q W_t(x,t)=0. \tag{48}$$

Therefore, W is independent of x [Eq. (47)]. The general solution of (48) is thus

$$W(x,t)=c\mathrm{e}^{-(t/\tau_q)} \tag{49}$$

with c as a constant. Applying the initial condition $W(x,0)=0$ yields

$$c=0. \tag{50}$$

Therefore,

$$W(x,t)\equiv 0 \tag{51}$$

i.e.,

$$T_1(x,t)\equiv T_2(x,t). \tag{52}$$

However, T_1 and T_2 are any two solutions of (4) so that we conclude that the solution of (4) is unique.

Similarly, we can also establish the uniqueness for the other combinations of boundary conditions.

2.4. Stability

We establish the stability with respect to the initial conditions in the following stability theorem.

Stability theorem. *If*

$$|\phi(x)|\leqslant\epsilon, \tag{53}$$

$$|\psi(x)|\leqslant\epsilon, \tag{54}$$

and

$$\left|\frac{\partial\phi}{\partial x}\right|\leqslant\epsilon, \tag{55}$$

the solution $T(x,t)$ *of* (4) *satisfies*

$$|T(x,t)|\leqslant C\epsilon. \tag{56}$$

Here, ϵ *is a small positive constant, and* C *is a nonnegative constant.*

Proof. For the case with the Robin condition at both $x=0$ and $x=l$, (43) yields, for (4) when $t>0$ and $t_0=0$,

$$
\begin{aligned}
&\alpha(\tau_T+\tau_q)[H_1T^2(0,t)+H_2T^2(l,t)]\\
&\quad+\int_0^l\{[T(x,t)+\tau_q T_t(x,t)]^2+\alpha(\tau_T+\tau_q)T_x^2(x,t)\}\,\mathrm{d}x\\
&\quad\leqslant\alpha(\tau_T+\tau_q)[H_1\phi^2(0)H_2\phi^2(l)]\\
&\quad+\int_0^l\{[\phi(x)+\tau_q\psi(x)]^2+\alpha(\tau_T+\tau_q)\phi_x^2(x)\}\,\mathrm{d}x\\
&\quad\leqslant\alpha(\tau_T+\tau_q)[H_1\epsilon^2+H_2\epsilon^2]\\
&\quad+\int_0^l\{[\epsilon+\tau_q\epsilon]^2+\alpha(\tau_T+\tau_q)\epsilon^2\}\,\mathrm{d}x=M\epsilon^2,
\end{aligned}
\tag{57}
$$

where Eqs. (53)–(55) have been used and

$$M=\alpha(\tau_T+\tau_q)(H_1+H_2)+[(1+\tau_q)^2+\alpha(\tau_T+\tau_q)]l. \tag{58}$$

This yields

$$\alpha(\tau_T+\tau_q)H_1T^2(0,t)\leqslant M\epsilon^2 \tag{59}$$

and

$$\int_0^l \alpha(\tau_T + \tau_q) T_x^2 \mathrm{d}x \leqslant M\epsilon^2, \tag{60}$$

which are equivalent to

$$|T(0,t)| \leqslant M_1 \epsilon \tag{61}$$

and

$$\int_0^l T_x^2(x,t)\,\mathrm{d}x \leqslant M_2 \epsilon^2. \tag{62}$$

Here,

$$M_1 = \sqrt{\frac{M}{\alpha(\tau_T + \tau_q)H_1}} \tag{63}$$

and

$$M_2 = \frac{M}{\alpha(\tau_T + \tau_q)}. \tag{64}$$

As [17],

$$\int_0^l |fg|\,\mathrm{d}x \leqslant \sqrt{\int_0^l f^2\,\mathrm{d}x}\sqrt{\int_0^l g^2\,\mathrm{d}x}$$

we have

$$\int_0^l |T_x(x,t)|\,\mathrm{d}x \leqslant \sqrt{\int_0^l T_x^2(x,t)\,\mathrm{d}x}\sqrt{\int_0^l l^2\,\mathrm{d}x}$$

$$= \sqrt{l\int_0^l T_x^2(x,t)\,\mathrm{d}x} \leqslant \sqrt{lM_2}\epsilon \tag{65}$$

in which (62) has been used.

Also,

$$T(x,t) = \int_0^x T_x(x,t)\,\mathrm{d}x + T(0,t). \tag{66}$$

Therefore,

$$|T(x,t)| \leqslant \int_0^x |T_x(x,t)|\,\mathrm{d}x + |T(0,t)|$$
$$\leqslant \sqrt{lM_2}\epsilon + M_1\epsilon = C\epsilon, \tag{67}$$

where

$$C = \sqrt{lM_2} + M_1. \tag{68}$$

Similarly, we can also prove the theorem for the other combinations of boundary conditions. It is interesting to note that (55) is also needed for the stability in addition to (53) and (54).

3. Solution structure

In this section, we develop two solution theorems expressing solutions of

$$\begin{cases} \frac{1}{\alpha}T_t(p,t) + \frac{\tau_q}{\alpha}T_{tt}(p,t) = \Delta T(p,t) \\ \quad +\tau_T \frac{\partial}{\partial t}\Delta T(p,t), \quad \Omega \times (0,+\infty), \\ L(T,T_n)|_{\partial\Omega} = 0, \\ T(M,0) = \phi(p), \quad T_t(p,0) = 0 \end{cases} \tag{69}$$

and

$$\begin{cases} \frac{1}{\alpha}T_t(p,t) + \frac{\tau_q}{\alpha}T_{tt}(p,t) = \Delta T(p,t) \\ \quad +\tau_T \frac{\partial}{\partial t}\Delta T(p,t) + f(p,t), \quad \Omega \times (0,+\infty), \\ L(T,T_n)|_{\partial\Omega} = 0, \\ T(p,0) = 0, \quad T_t(p,0) = 0 \end{cases} \tag{70}$$

in terms of the solution of

$$\begin{cases} \frac{1}{\alpha}T_t(p,t) + \frac{\tau_q}{\alpha}T_{tt}(p,t) = \Delta T(p,t) \\ \quad +\tau_T \frac{\partial}{\partial t}\Delta T(p,t), \quad \Omega \times (0,+\infty), \\ L(T,T_n)|_{\partial\Omega} = 0, \\ T(p,0) = 0, T_t(p,0) = \psi(p). \end{cases} \tag{71}$$

Here, p denotes a point in the space domain Ω with the boundary $\partial\Omega$, Δ the Laplacian, T_n the normal derivative of T, $L(T,T_n)$ represents linear functions of T and T_n, and $L(T,T_\mathrm{n})|_{\partial\Omega} = 0$ denotes homogeneous boundary conditions. Note that commonly used Dirichlet, Neumann and Robin boundary conditions are the special cases of the linear function L. We limit the present work to the case that f, ϕ and ψ satisfy conditions for well-posedness and that the order of differentiation is interchangeable for some high-order partial derivatives of T with respect to the time and spatial coordinates.

Theorem 1. *Let $W(\psi,p,t)$ denote the solution of* (71). *The solution of* (69) *can be written as*

$$T_1(p,t) = \frac{1}{\tau_q}\left[W(\phi,p,t) + \tau_q\frac{\partial W(\phi,p,t)}{\partial t} + W(\phi_1,p,t)\right], \tag{72}$$

where

$$\phi_1 \equiv -\alpha\tau_T\Delta\phi(p). \tag{73}$$

Proof. As $W(\psi,p,t)$ is the solution of (71), we have

$$\begin{cases} \frac{1}{\alpha}\frac{\partial}{\partial t}W(\phi,p,t) + \frac{\tau_q}{\alpha}\frac{\partial^2}{\partial t^2}W(\phi,p,t), \quad \Omega\times(0,+\infty), \\ -\Delta W(\phi,p,t) - \tau_T\frac{\partial}{\partial t}\Delta W(\phi,p,t) = 0, \\ L\left[W(\phi,p,t), \frac{\partial}{\partial n}W(\phi,p,t)\right]\big|_{\partial\Omega} = 0, \\ W(\phi,p,0) = 0, \quad \frac{\partial}{\partial t}W(\phi,p,0) = \phi(p) \end{cases} \tag{74}$$

and

$$\begin{cases} \frac{1}{\alpha}\frac{\partial}{\partial t}W(\phi_1,p,t) + \frac{\tau_q}{\alpha}\frac{\partial^2}{\partial t^2}W(\phi_1,p,t), \quad \Omega\times(0,+\infty) \\ -\Delta W(\phi_1,p,t) - \tau_T\frac{\partial}{\partial t}\Delta W(\phi_1,p,t) = 0, \\ L\left[W(\phi_1,p,t), \frac{\partial}{\partial n}W(\phi_1,p,t)\right]\big|_{\partial\Omega} = 0, \\ W(\phi_1,p,0) = 0, \quad \frac{\partial}{\partial t}W(\phi_1,p,0) = \phi_1(p). \end{cases} \tag{75}$$

Hence

$$\frac{1}{\alpha}\frac{\partial}{\partial t}T_1 + \frac{\tau_q}{\alpha}\frac{\partial^2}{\partial t^2}T_1 - \Delta T_1 - \tau_T\frac{\partial}{\partial t}\Delta T_1$$
$$= \frac{1}{\tau_q}\left[\frac{1}{\alpha}\frac{\partial W(\phi,p,t)}{\partial t} + \frac{\tau_T}{\alpha}\frac{\partial^2 W(\phi,p,t)}{\partial t^2} - \Delta W(\phi,p,t) - \tau_T\frac{\partial}{\partial t}\Delta W(\phi,p,t)\right] + \frac{\partial}{\partial t}\left[\frac{1}{\alpha}\frac{\partial W(\phi,p,t)}{\partial t} + \frac{\tau_q}{\alpha}\frac{\partial^2 W(\phi,p,t)}{\partial t^2} - \Delta W(\phi,p,t) - \tau_T\frac{\partial}{\partial t}\Delta W(\phi,p,t)\right]$$
$$+ \frac{1}{\tau_q}\left[\frac{1}{\alpha}\frac{\partial W(\phi_1,p,t)}{\partial t} + \frac{\tau_q}{\alpha}\frac{\partial^2 W(\phi_1,p,t)}{\partial t^2} - \Delta W(\phi_1,p,t) - \tau_T\frac{\partial}{\partial t}\Delta W(\phi_1,p,t)\right] = 0$$

which indicates that the T_1 in (72) satisfies the equation in (69).

Also,

$$L\left(T_1, \frac{\partial}{\partial n}T_1\right) = L\left\{\frac{1}{\tau_q}\left[W(\phi,p,t) + \tau_q\frac{\partial W(\phi,p,t)}{\partial t} + W(\phi_1,p,t)\right], \frac{1}{\tau_q}\frac{\partial}{\partial n}\left[W(\phi,p,t) + \tau_q\frac{\partial W(\phi,p,t)}{\partial t} + W(\phi_1,p,t)\right]\right\}$$
$$= \frac{1}{\tau_q}L\left[W(\phi,p,t), \frac{\partial}{\partial n}W(\phi,p,t)\right] + \frac{\partial}{\partial t}L\left[W(\phi,p,t), \frac{\partial}{\partial n}W(\phi,p,t)\right] + \frac{1}{\tau_q}L\left[W(\phi_1,p,t), \frac{\partial}{\partial n}W(\phi_1,p,t)\right]$$

and

$$L\left(T_1, \frac{\partial}{\partial n}T_1\right)\Big|_{\partial\Omega} = \frac{1}{\tau_q}L\left[W(\phi,p,t), \frac{\partial}{\partial n}W(\phi,p,t)\right]\Big|_{\partial\Omega} + \frac{\partial}{\partial t}L\left[W(\phi,p,t), \frac{\partial}{\partial n}W(\phi,p,t)\right]\Big|_{\partial\Omega} + \frac{1}{\tau_q}L\left[W(\phi_1,p,t), \frac{\partial}{\partial n}W(\phi_1,p,t)\right]\Big|_{\partial\Omega} = 0$$

in which boundary conditions in (74) and (75) have been used. This indicates that the T_1 in (72) satisfies the boundary condition in (69).

Finally, by (74) and (75),

$$T_1(p,0) = \frac{1}{\tau_q}\left[W(\phi,p,t) + \tau_q\frac{\partial W(\phi,p,t)}{\partial t} + W(\phi_1,p,t)\right]\Big|_{t=0} = \frac{1}{\tau_q}\left(W(\phi,p,0) + \tau_q\frac{\partial W(\phi,p,0)}{\partial t} + W(\phi_1,p,0)\right) = \frac{\partial W(\phi,p,0)}{\partial t} = \phi$$

and

$$\frac{\partial}{\partial t}T_1(p,t)|_{t=0} = \frac{1}{\tau_q}\frac{\partial}{\partial t}\left[W(\phi,p,t) + \tau_q\frac{\partial W(\phi,p,t)}{\partial t} + W(\phi_1,p,t)\right]\Big|_{t=0}$$
$$= \frac{1}{\tau_q}\frac{\partial}{\partial t}\left[W(\phi,p,t) + \tau_q\frac{\partial W(\phi,p,t)}{\partial t}\right]\Big|_{t=0} + \frac{\phi_1(p)}{\tau_q}$$
$$= \frac{\alpha}{\tau_q}\left[\Delta W(\phi,p,t) + \tau_T\frac{\partial}{\partial t}(\Delta W(\phi,p,t))\right]\Big|_{t=0} + \frac{\phi_1(p)}{\tau_q}$$
$$= \alpha\frac{\tau_T}{\tau_q}\Delta\phi(p) + \frac{1}{\tau_q}\phi_1(p) = 0.$$

Therefore, the T_1 in (72) also satisfies initial conditions in (69).

Theorem 2. *Let $W(\psi,p,t)$ denote the solution of* (71). *The solution of* (70) *can be written as*

$$T_2(p,t) = \int_0^t W(f_\tau,p,t-\tau)\,d\tau, \tag{76}$$

where

$$f_\tau = \frac{\alpha}{\tau_q}f(p,\tau). \tag{77}$$

Proof. As $W(\psi,p,t)$ is the solution of (71), we have

$$\begin{cases} \frac{1}{\alpha}\frac{\partial}{\partial t}W(f_\tau,p,t-\tau) + \frac{\tau_q}{\alpha}\frac{\partial^2}{\partial t^2}W(f_\tau,p,t-\tau), \quad \Omega\times(0,+\infty), \\ -\Delta W(f_\tau,p,t-\tau) - \tau_T\frac{\partial}{\partial t}\Delta W(f_\tau,p,t-\tau) = 0, \\ L\left[W(f_\tau,p,t-\tau), \frac{\partial}{\partial n}W(f_\tau,p,t-\tau)\right]\big|_{\partial\Omega} = 0, \\ W(f_\tau,p,t-\tau)|_{t=\tau} = 0, \frac{\partial}{\partial t}W(f_\tau,p,t-\tau)\,|_{t=\tau} = \frac{\alpha}{\tau_q}f(p,\tau). \end{cases} \tag{78}$$

Therefore,

$$\frac{1}{\alpha}\frac{\partial}{\partial t}T_2 + \frac{\tau_q}{\alpha}\frac{\partial^2}{\partial t^2}T_2 - \Delta T_2 - \tau_T\frac{\partial}{\partial t}\Delta T_2$$
$$= \frac{1}{\alpha}\frac{\partial}{\partial t}\int_0^t W(f_\tau,p,t-\tau)\,d\tau + \frac{\tau_q}{\alpha}\frac{\partial^2}{\partial t^2}\int_0^t W(f_\tau,p,t-\tau) \times d\tau - \Delta\int_0^t W(f_\tau,p,t-\tau)\,d\tau - \tau_T \times \frac{\partial}{\partial t}\Delta\int_0^t W(f_\tau,p,t-\tau)\,d\tau$$
$$= \frac{1}{\alpha}\left[\int_0^t \frac{\partial W(f_\tau,p,t-\tau)}{\partial t}d\tau + W(f_\tau,p,t-\tau)|_{\tau=t}\right] + \frac{\tau_q}{\alpha}\left[\int_0^t \frac{\partial^2 W(f_\tau,p,t-\tau)}{\partial t^2} + \frac{\partial W(f_\tau,p,t-\tau)}{\partial t}\Big|_{\tau=t}\right]$$
$$- \Delta\int_0^t W(f_\tau,p,t-\tau)\,d\tau - \tau_T\Delta\frac{\partial}{\partial t}\int_0^t W(f_\tau,p,t-\tau)\,d\tau$$

$$= \int_0^t \frac{1}{\alpha} \frac{\partial W(f_\tau, p, t-\tau)}{\partial t}\, \mathrm{d}\tau + \int_0^t \frac{\tau_q}{\alpha} \times \frac{\partial^2 W(f_\tau, p, t-\tau)}{\partial t^2}\, \mathrm{d}\tau + f(p,t) - \int_0^t \Delta W(f_\tau, p, t-\tau)\, \mathrm{d}\tau - \int_0^t \tau_T \Delta \frac{\partial W(f_\tau, p, t-\tau)}{\partial t}\, \mathrm{d}\tau = f(p,t)$$

which indicates that the T_2 in (76) satisfies the equation in (70).

Also,

$$L\left(T_2, \frac{\partial}{\partial n} T_2\right)\Big|_{\partial\Omega} = L\left[\int_0^t W(f_\tau, p, t-\tau)\, \mathrm{d}\tau, \frac{\partial}{\partial n}\int_0^t W(f_\tau, p, t-\tau)\, \mathrm{d}\tau\right]\Bigg|_{\partial\Omega}$$
$$= L\left[\int_0^t W(f_\tau, p, t.-\tau)\, \mathrm{d}\tau, \int_0^t \frac{\partial}{\partial n} W(f_\tau, p, t-\tau)\, \mathrm{d}\tau\right]\Bigg|_{\partial\Omega}$$
$$= \int_0^t L\left[W(f_\tau, p, t-\tau), \frac{\partial}{\partial n} W(f_\tau, p, t-\tau)\right]\Bigg|_{\partial\Omega} \mathrm{d}\tau = 0$$

in which the boundary condition in (78) has been used. Therefore, the T_2 in (76) satisfies the boundary condition in (70).

Finally,

$$T_2(p,0) = \int_0^0 W(f_\tau, p, t-\tau)\, \mathrm{d}\tau = 0$$

and, by (78)

$$\frac{\partial}{\partial t} T_2(p,t)|_{t=0} = \left[\int_0^t \frac{\partial}{\partial t} W(f_\tau, p, t-\tau)\mathrm{d}\tau + W(f_\tau, p, t-\tau)|_{\tau=t}\right]\Bigg|_{t=0} = 0.$$

Therefore, the T_2 in (76) also satisfies initial conditions in (70).

By Theorems 1 and 2 and the principle of superposition, we can express the solution $T(p,t)$ of

$$\begin{cases} \frac{1}{\alpha} T_t(p,t) + \frac{\tau_q}{\alpha} T_{tt}(p,t) = \Delta T(p,t) \\ \quad + \tau_T \frac{\partial}{\partial t} \Delta T(p,t) + f(p,t), \quad \Omega \times (0, +\infty) \\ L(T, T_n)|_{\partial\Omega} = 0, \\ T(p,0) = \phi(p), T_t(p,0) = \psi(p) \end{cases} \tag{79}$$

in term of W as

$$T(p,t) = W(\psi, p, t) + \frac{1}{\tau_q}\left[W(\phi, p, t) + \tau_q \frac{\partial W(\phi, p, t)}{\partial t} + W(\phi_1, p, t)\right] + \int_0^t W(f_\tau, p, t-\tau)\, \mathrm{d}\tau \tag{80}$$

with ϕ_1 and f_τ defined by Eqs. (73) and (77).

4. Concluding remarks

The well-posedness is examined for 1D dual-phase-lagging heat conduction equations under Dirichlet, Neumann or Robin boundary conditions. The method of separation of variables is used to find a solution. The inequality (43) developed in the present work is employed to establish its uniqueness and stability with respect to initial conditions. This is of fundamental importance for using dual-phase-lagging heat conduction equations in microscale heat conduction.

Two solution structure theorems are developed for dual-phase-lagging heat conduction equations under linear boundary conditions. Contributions (to the temperature field) of the initial temperature distribution and the source term are shown to be expressible by that of the initial time-rate change of the temperature. This reveals the solution structure and significantly simplifies the development of solutions of dual-phase-lagging heat conduction equations.

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